On Endo-trivial Modules for p-Solvable Groups

Gabriel Navarro
Departament d'Àlgebra
Facultat de Matemàtiques
Universitat de València
46100 Burjassot
València
SPAIN
E-mail: gabriel.navarro@uv.es
&
Geoffrey R. Robinson,
Institute of Mathematics
University of Aberdeen
Aberdeen
AB24 3UE
SCOTLAND
E-mail: g.r.robinson@abdn.ac.uk
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Introduction: In this note, we will prove a conjecture of J. Carlson, N. Mazza and J. Thévenaz [1], namely, we will prove that if G is a finite p-nilpotent group which contains a non-cyclic elementary Abelian p-subgroup and k is an algebraically closed field of characteristic p, then all simple endo-trivial kG-modules are 1-dimensional. In fact, we do rather more: we prove the analogous result directly in the case that G is p-solvable and contains an elementary Abelian p-subgroup of order p^2. Carlson, Mazza and Thévenaz had reduced the proof of this result for p-solvable G to the p-nilpotent case, (and had proved the result in the solvable case), but our method is somewhat different. Our proof does require the classification of finite simple groups. Specifically, we require the well-known fact that the outer automorphism group of a finite simple group of order prime to p has cyclic Sylow p-subgroups (see, for example, Theorem 7.1.2 of Gorenstein, Lyons and Solomon, [3]).

Let us recall that a kG-module M is endo-trivial if $M \otimes M^* \cong k \oplus N$, where N is a projective kG-module. If $|G|$ is divisible by p, any endo-trivial kG-module has dimension prime to p. The vertex of any indecomposable endo-trivial kG-module is a Sylow p-subgroup of G. We remark that if M is an endo-trivial kG-module which is not 1-dimensional, then a Sylow p-subgroup of G acts faithfully on $M \otimes M^*$ and hence acts faithfully on M.

In the first Lemma, we summarize some properties of endo-trivial modules which are for the most part well-known. We recall that a subgroup H of a finite group G is said to be strongly p-embedded if p divides $|H|$ and p does not divide $|H \cap H^g|$ for each $g \in G \backslash H$.

Lemma 1: *Let M be an endo-trivial module for a finite group X. Then:*
i) Every non-projective summand of $\mathrm{Res}^X_Z(M)$ is endo-trivial for each subgroup

Z of X.
ii) M has a unique non-projective indecomposable summand which is itself endo-trivial.
iii) If $M \cong U \otimes V$ for kX-modules U and V, then both U and V are endo-trivial.
iv) If $M \cong \mathrm{Ind}_Y^X(L)$ for some proper subgroup Y of X, and some kY-module L, then L is endo-trivial and Y is strongly p-embedded in X.

PROOF: The first two parts are clear. To prove iii), notice that U and V each have dimension prime to p. Hence $U \otimes U^* = k \oplus S$ and $V \otimes V^* = k \oplus T$, where S and T are kX-modules. Then $M \otimes M^* = k \oplus S \oplus T \oplus (S \otimes T)$, so that S and T must both be projective.

iv) Notice that L is isomorphic to a non-projective direct summand of $\mathrm{Res}_Y^X(M)$ in this case, so that L is endo-trivial. Since M has dimension prime to p, we see that Y must contain a Sylow p-subgroup of X.

Now

$$M \otimes M^* \cong \mathrm{Ind}_Y^X[L \otimes \mathrm{Res}_Y^X(M^*)],$$

so that $M \otimes M^*$ has a direct summand isomorphic to $\mathrm{Ind}_Y^X(L \otimes L^*)$, and in particular, a direct summand isomorphic to $\mathrm{Ind}_Y^X(k)$. Since M is endo-trivial, this implies that the only non-projective indecomposable summand of $\mathrm{Ind}_Y^X(k)$ is k. By the Mackey formula (applied to the restriction of this permutation module to Y) this implies that $\mathrm{Ind}_{Y \cap Y^x}^Y(k)$ is projective for each $x \in X \backslash Y$, so that $Y \cap Y^x$ is a p'-subgroup for each $x \in X \backslash Y$ and Y is strongly p-embedded in X.

Remark: The converse of part iv) is also true: if an endo-trivial kY-module is induced from the strongly p-embedded subgroup Y of X, then the resulting kX-module is also endo-trivial. This is almost immediate from the proof of iv) above and Mackey's Theorem.

Corollary: *Let X be a p-solvable finite group containing an elementary Abelian subgroup of order p^2. Then no endo-trivial kX-module is induced from a proper subgroup of X.*

Proof: Let Q be a Sylow p-subgroup of X. If such a module were induced from a proper subgroup Y of X, then Y would be strongly p-embedded in X (and may be chosen to contain Q) by the previous Lemma. Let $Z = O_{p',p}(X)$. Then $X = O_{p'}(X)N_X(Q \cap Z)$. Now, as Q contains an elementary Abelian subgroup of order p^2, by 6.2.4 of Gorenstein, [2], for example, we have

$$O_{p'}(X) \leq \langle C_X(u) : u \in Q^\# \rangle \leq Y,$$

and we also have $N_X(Q \cap Z) \leq Y$. Hence $X \leq Y$, contrary to the fact that Y is proper.

The following Lemma is well-known, but we include its proof for completeness:

Lemma 2: *Let $G = \langle x \rangle N$ be a finite p-nilpotent group with Sylow p-subgroup $\langle x \rangle$ of order p and normal p-complement N. Suppose that V is a simple endo-trivial kG-module, and let W be its Green correspondent for $N_G(\langle x \rangle)(= C_G(x))$. Then all indecomposable summands of $\mathrm{Res}^{C_G(x)}_{C_N(x)}(W)$ are isomorphic and 1-dimensional.*

Proof: Since V has dimension prime to p, the restriction of V to N is simple. Notice that $C_G(x) = \langle x \rangle \times C_N(x)$, and that every indecomposable $kC_G(x)$-module is expressible as a tensor product $A \otimes B$ where $C_N(x)$ acts trivially on A and x acts indecomposably on A, while x acts trivially on B and $C_N(x)$ acts irreducibly on B.

Now W is indecomposable, and is also endo-trivial. Writing W in the above fashion as $A \otimes B$, both A and B are endo-trivial by part iii) of Lemma 1. Then B is 1-dimensional, since $\langle x \rangle$ acts trivially on B. (Notice also that $\dim_k(A) \leq p$, so we either have $\dim_k(A) = p - 1$ or $\dim_k(A) = 1$).

We recall that a component of a finite group X is a subnormal quasi-simple subgroup of X. Distinct components of X centralize each other, and all components of X centralize the Fitting subgroup $F(X)$. The central product of the components of X is denoted by $E(X)$. The following Lemma is probably well-known, but we include a proof.

Lemma 3: *Let G be a perfect finite group with $G = E(G)$ and with $Z = Z(G)$ a cyclic p'-group. Suppose that $G \lhd H$ for some finite group H with $Z \leq Z(H)$. Suppose further that the components of G are all conjugate within H and that the element x of order p in H permutes the components of G semi-regularly by conjugation. Then $C_G(x)$ is isomorphic to a central product of components of G, one from each $\langle x \rangle$-orbit. In particular, $C_G(x)$ is perfect.*

Proof: We first note that Z is contained in each component of G. For suppose that L is a component of G and $W = L \cap Z < Z$. Then all components of G/W are simple, and G/W has a non-trivial Abelian direct factor Z/W, so is not perfect, a contradiction. Let $L_1, L_2, \ldots, L_n$ be representatives for the $\langle x \rangle$-orbits of components of G. Let $T = L_1 L_2 \ldots L_n$, a central product of mutually centralizing components. Then $T, T^x, \ldots, T^{x^{p-1}}$ are mutually centralizing, since no two of them contain a common component. We may thus define a homomorphism $\phi : T \to C_G(x)$ via $t\phi = tt^x \ldots t^{x^{p-1}}$. In the case that $Z = 1$, this is clearly a surjection. When $Z \neq 1$, we have $Z \leq T\phi$ since x acts trivially on Z and Z is a p'-group. Also, we have (by, for example, 5.3.15 of Gorenstein [2]) $C_{H/Z}(xZ) = C_H(x)/Z$. The analysis in the $Z = 1$ applies to G/Z, so that $C_{G/Z}(xZ)$ is clearly isomorphic to T/Z and hence $C_G(x)$ is isomorphic to T since T injects into $C_G(x)$.

Theorem: *Let X be a p-solvable group which contains an elementary Abelian subgroup Q of order p^2 and let V be a simple endo-trivial kX-module. Then V is 1-dimensional.*

Proof: If possible, choose a counterexample (X, V) so that first $\dim_k(V)$, then $|X|$, are minimized. Then V is a faithful kP-module, where P is a Sylow p-

subgroup of G. But V is simple, so that $O_p(X)$ acts trivially on V. Hence $O_p(X) = 1$. More generally, the kernel of the action of X on V is a p'-group, so that V is a faithful kX-module by minimality. We know that V is a primitive kX-module by the Corollary above. This enables us to perform standard Clifford-theoretic reductions, and the endo-trivial condition turns out to be compatible with these reductions.

Let Y be a normal subgroup of X minimal subject to strictly containing $Z(X)$. Since $Y/Z(X)$ is a minimal normal subgroup of $X/Z(X)$, we know that $Y/Z(X)$ is a direct product of simple groups. If $Y/Z(X)$ is Abelian, then $Y' \leq Z(X)$ and Y is nilpotent. Notice that $Y/Z(X)$ is not a p-group as $O_p(X) = 1$. Let U be an irreducible summand of $\mathrm{Res}^X_Y(V)$. Since V is primitive and Y is non-central, the isomorphism type of U is X-stable, but U is not 1-dimensional. The usual Clifford-theoretic construction yields a p'-central extension $\hat{X}$ of X such that U extends to a simple $k\hat{X}$-module, and such that $V \cong U \otimes W$ as $k\hat{X}$-module (in fact $W \cong \mathrm{Hom}_{\hat{Y}}(U, V)$). By part iii) of Lemma 1, both U and W are endo-trivial as $k\hat{X}$-modules, since V is also endotrivial as $k\hat{X}$-module (for $\hat{X}$ acts as X does on $V \otimes V^*$). The Sylow p-subgroups of X and of $\hat{X}$ are clearly isomorphic. If neither U nor W is one dimensional, we have a contradiction to the minimal choice of (X, V). Hence W must be one dimensional, as U is not.

Hence $\dim_k(U) = \dim_k(V)$, so that V restricts irreducibly to Y. Hence $Z(Y) \leq Z(X) = C_X(Y)$ by Schur's Lemma. Now Y is either nilpotent of class 2 or else is the central product of $Z(X)$ with a single conjugacy class of components, each of order prime to p. By the minimal choice of (X, V), we now have $X = YQ$. If Y is not nilpotent of class 2, then $Y' = E(Y)$ still acts irreducibly on V, so the minimal choice of (X, V) gives $X = E(Y)Q = E(X)Q$ in that case.

Suppose that Y is nilpotent. This case was deal with by Carlson, Mazza and Thévenaz in [1], but we provide a different argument to dispose of it. We know that $Y/Z(Y)$ is a minimal normal subgroup of $X/Z(X)$. For any $a \in Q^{\#}$, we have $Z(Y) \leq C_Y(a) \lhd Y$. Furthermore, $C_Y(a)$ is Q-invariant, so $C_Y(a) \lhd YQ = X$. However, $C_Y(a) \neq Y$, as Y acts irreducibly on V and a has order p. Thus $C_Y(a) = Z(Y)$. Since a was arbitrary, and $Z(Y) \neq Y = \langle C_Y(a) : a \in Q^{\#} \rangle$, this is a contradiction. Hence $Y = E(Y)$.

We have already remarked that Q acts faithfully on V. Hence no element of Q can centralize Y, as Y acts irreducibly on V. Since Y is a p'-group and Q centralizes $Z(Y)$, the action of Q on $Y/Z(Y)$ is faithful. Hence Y is not quasi-simple, for otherwise the outer automorphism group of $Y/Z(Y)$ has cyclic Sylow p-subgroups. Since $Y/Z(Y)$ is a minimal normal subgroup of $X/Z(X)$, the components of Y are transitively permuted under conjugation by Q. Hence there is an element $a \in Q$ which acts semi-regularly by conjugation on the components of Y. Then $C_Y(a)$ is perfect by Lemma 3. In particular, there is no non-trivial 1-dimensional simple $kC_Y(a)$-module.

However, by Lemma 2, the Green correspondent of $\mathrm{Res}^X_{Y\langle a \rangle}(V)$ for $C_X(a)$ lies over a 1-dimensional module for $C_Y(a)$, so lies over the trivial module of

$C_Y(a)$. Hence that Green correspondent lies in the principal block of $C_X(a)$. By Brauer's Third Theorem (and the compatibility between Green correspondence and Brauer correspondence), $\mathrm{Res}^X_{Y\langle a\rangle}(V)$ lies in the principal block of $kY\langle a\rangle$, a contradiction, as Y acts faithfully on V and $Y = O_{p'}(Y\langle a\rangle)$.

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